

SOME ALTERNATIVE DEFINITIONS FOR THE "PLUS-MINUS" INTERPOLATION SPACES $\langle A_0, A_1 \rangle_\theta$ OF JAAK PEETRE

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ABSTRACT. The Peetre "plus-minus" interpolation spaces $\langle A_0, A_1 \rangle_\theta$ are defined via conditions regarding the unconditional convergence of Banach space valued series of the form $\sum_{n=-\infty}^{\infty} 2^{(j-\theta)n} a_n$ or, alternatively, $\sum_{n=-\infty}^{\infty} e^{(j-\theta)n} a_n$, for $j \in \{0, 1\}$. It may seem intuitively obvious that using powers of 2 or of e , or powers of some other constant number greater than 1 in these definitions should produce the same spaces to within equivalence of norms. To allay any doubts, we here offer an explicit proof of this fact, via a "continuous" definition of the same spaces, where integrals replace the above series. This apparently new definition, which is also in some sense a "limiting case" of the usual "discrete" definitions, may be relevant in the study of the connection between $\langle A_0, A_1 \rangle_\theta$ and the Calderón complex interpolation space $[A_0, A_1]_\theta$ in the case where (A_0, A_1) is a couple of Banach lattices. Related results can probably be obtained for the Gustavsson-Peetre variant of the "plus-minus" spaces.

1. "PRE-INTRODUCTION" - SOME BACKGROUND

In [14], among a number of other very interesting things, Jaak Peetre introduced several kinds of interpolation spaces, including ones which we will soon define in Section 2, and which he denoted by $\langle A_0, A_1 \rangle_\theta$. (The bibliography of [14] indicates that such spaces, or some variants of them, had apparently already been considered as early as in 1962 and were discussed in (Swedish) lecture notes written jointly at that time by Peetre and Arne Persson, for a course at the University of Lund.)

It should be mentioned that, a decade after the appearance of [14], the remarkable paper [10] of Svante Janson identified $\langle A_0, A_1 \rangle_\theta$ as an "orbit" space and offered other additional insights about it (and about several other kinds of interpolation spaces). Continuing in the spirit of [14], the two papers [10] and [7] each exhibited different ways of fitting the spaces $\langle A_0, A_1 \rangle_\theta$ and other kinds of interpolation spaces into more general frameworks. (The approach of [10] is elaborated upon further in [13], and some further discussion of the contents of [7] can be found in [8].) But the only result from these papers which I will use here is a counterexample from [10] which will be mentioned below in Remark 4.2. Also I will not deal with the variants $\langle A_0, A_1, \rho \rangle$ of the spaces $\langle A_0, A_1 \rangle_\theta$ which were introduced and studied in [9] by Jan Gustavsson and Peetre and also later in [10]. The contents of this note can surely be adapted to give similar results about the spaces $\langle A_0, A_1, \rho \rangle$, in particular when the function parameter ρ in their definition is of the form $\rho(t) = t^\theta$ for some $\theta \in (0, 1)$.

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The contents of the rest of this note are as follows: In Section 2 we recall the various slightly different "discrete" definitions of the spaces $\langle A_0, A_1 \rangle_\theta$, i.e., the definitions which have been used to date, and which have been implicitly assumed to be equivalent to each other. We also mention some of the basic properties of these spaces and of the auxiliary spaces which are used to define them. In Section 3 we introduce a new "continuous" definition of $\langle A_0, A_1 \rangle_\theta$ and show that it is equivalent to each of the "discrete" definitions of Section 2, and therefore we also confirm that those definitions are indeed equivalent to each other. Section 4 could be considered to be a sort of appendix. In it we recall the well known and obvious fact that the space $\langle A_0, A_1 \rangle_\theta$ is contained in the complex interpolation space $[A_0, A_1]_\theta$, and note that this is a norm one inclusion. Section 5 is most definitely an appendix, which describes straightforward proofs of the completeness of $\langle A_0, A_1 \rangle_\theta$ and of the auxiliary space used to define it, for the convenience of any reader who is less familiar with these matters.

2. MORE INTRODUCTION - THE STANDARD "DISCRETE" DEFINITION(S) OF THE SPACE $\langle A_0, A_1 \rangle_\theta$, AND SOME OF ITS PROPERTIES

We recall that a two-sided series $\sum_{n \in \mathbb{Z}} a_n$ of elements a_n in a Banach space A is said to be *unconditionally convergent* if, for every bounded sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ of scalars, the sequence of partial sums $\left\{ \sum_{n=-N}^N \lambda_n a_n \right\}_{N \in \mathbb{N}}$ converges in norm in A as N tends to ∞ . This property readily implies, and is therefore equivalent to, the following stronger condition:

$$(2.1) \quad \lim_{N \rightarrow \infty} \sup \left\{ \left\| \sum_{|n| \geq N} \lambda_n a_n \right\|_A : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} = 0,$$

where Λ is the set of all scalar sequences $\{\lambda_k\}_{k \in \mathbb{Z}}$ which satisfy $\sup_{k \in \mathbb{Z}} |\lambda_k| \leq 1$. (The straightforward proof that unconditional convergence implies (2.1) is a simpler variant of the proof below of Fact 3.3.)

Of course (2.1) implies that

$$(2.2) \quad \sup \left\{ \left\| \sum_{n=-\infty}^{\infty} \lambda_n a_n \right\|_A : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} < \infty.$$

Remark 2.1. Note that, in seeming contrast to some phenomena in some other more elementary situations, the existence of the limits $\lim_{N \rightarrow \infty} \sum_{n=-N}^N \lambda_n a_n$ for all $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda$ is obviously equivalent to the existence of both of the limits $\lim_{N \rightarrow \infty} \sum_{n=1}^N \lambda_n a_n$ and $\lim_{N \rightarrow \infty} \sum_{n=-N}^0 \lambda_n a_n$ for all $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda$.

Let \mathbb{F} be either \mathbb{R} or \mathbb{C} and let (A_0, A_1) be an arbitrary Banach couple where \mathbb{F} is the field of scalars for A_0 and also for A_1 . For each choice of the constants $r > 0$ and $\theta \in (0, 1)$, let $J(\theta, r, A_0, A_1)$ denote the space of $A_0 \cap A_1$ valued sequences $\{a_n\}_{n \in \mathbb{Z}}$ having the two properties that $\sum_{n \in \mathbb{Z}} e^{-r\theta n} a_n$ is an unconditionally convergent series in A_0 and $\sum_{n \in \mathbb{Z}} e^{r(1-\theta)n} a_n$ is an unconditionally convergent series in A_1 . In view of the

properties of unconditionally convergent series discussed above, we see (cf. (2.2)) that the quantity

$$(2.3) \quad \left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, A_0, A_1)} := \sup \left\{ \left\| \sum_{n=-\infty}^{\infty} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : j \in \{0, 1\}, \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}$$

is finite for every sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $J(\theta, r, A_0, A_1)$ and therefore defines a norm on that space.

Remark 2.2. It is clear that, for each $\{a_n\}_{n \in \mathbb{Z}}$ in $J(\theta, r, A_0, A_1)$, the value of the supremum in (2.3) will not change if we replace Λ in (2.3) by its subset Λ_0 consisting of those sequences which have only finitely many non-zero elements.

Remark 2.3. We note that, for $j \in \{0, 1\}$ and every sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $J(\theta, r, A_0, A_1)$,

$$(2.4) \quad \sup \left\{ \left\| \sum_{n \in U} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} \leq \sup \left\{ \left\| \sum_{n \in V} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}$$

whenever U and V are (finite or infinite) sets which satisfy $U \subset V \subset \mathbb{Z}$. This follows from the simple fact that the left side of (2.4) equals

$$\sup \left\{ \left\| \sum_{n \in V} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda_U \right\},$$

where Λ_U is the subset of Λ consisting of all those sequences $\{\lambda_k\}_{k \in \mathbb{Z}}$ which satisfy $\lambda_k = 0$ for all $k \in \mathbb{Z} \setminus U$.

Remark 2.4. For our convenience in some later parts of this discussion, let us here explicitly state that, for every $A_0 \cap A_1$ valued sequence $\{a_n\}_{n \in \mathbb{N}}$, the following three conditions are equivalent:

- (i) $\{a_n\}_{n \in \mathbb{Z}}$ is an element of $J(\theta, r, A_0, A_1)$.
- (ii) For $j \in \{0, 1\}$ and for each $\varepsilon > 0$ and each $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda$, there exists a positive integer $N(\varepsilon)$ (which could in principle also depend on $\{\lambda_k\}_{k \in \mathbb{Z}}$ and j) such that $\left\| \sum_{n_1 \leq |n| \leq n_2} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} \leq \varepsilon$ whenever $N(\varepsilon) \leq n_1 < n_2$.
- (iii) For each $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that

$$\left\| \sum_{n_1 \leq |n| \leq n_2} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} \leq \varepsilon$$

for every $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda$ and for $j \in \{0, 1\}$ whenever $N(\varepsilon) \leq n_1 < n_2$.

The equivalence of (i) and (ii) and the implication (iii) \Rightarrow (ii) are obvious. The implication (i) \Rightarrow (iii) follows immediately from the remarks at the beginning of this section concerning the condition (2.1), i.e., from a straightforward proof similar to that of Fact 3.3.

It is very easy to check (via reasoning similar to that which appears below in Fact 3.5) that, for each $\{a_n\}_{n \in \mathbb{Z}} \in J(\theta, r, A_0, A_1)$, and each constant integer k_0 , the partial sums $\sum_{n=-N}^{N+k_0} a_n$ converge in the norm of $A_0 + A_1$, to a limit which does not depend on k_0 and which we will denote by $\sum_{n=-\infty}^{\infty} a_n$, and, furthermore, that this limit satisfies

$$(2.5) \quad \left\| \sum_{n=-\infty}^{\infty} a_n \right\|_{A_0 + A_1} \leq 2 \left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, A_0, A_1)}.$$

Let us now define the space $\langle A_0, A_1 \rangle_{\theta, (r)}$ to consist of all elements $a \in A_0 + A_1$ which can be represented in the form

$$(2.6) \quad a = \sum_{n=-\infty}^{\infty} a_n$$

for some sequence $\{a_n\}_{n \in \mathbb{Z}} \in J(\theta, r, A_0, A_1)$. We shall norm this space by

$$(2.7) \quad \|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} := \inf \left\{ \left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, A_0, A_1)} \right\},$$

where the infimum is taken over all sequences $\{a_n\}_{n \in \mathbb{Z}} \in J(\theta, r, A_0, A_1)$ which satisfy (2.6). We can use (2.5) to show that (2.7) defines a norm, rather than merely a semi-norm, and also to show that $\langle A_0, A_1 \rangle_{\theta, (r)}$ is continuously embedded into $A_0 + A_1$.

If we choose $r = \ln 2$, then the above definition of $\langle A_0, A_1 \rangle_{\theta, (r)}$ coincides exactly with the definition of the space denoted by $\langle A_0, A_1 \rangle_\theta$ on lines 9–10 of [14, p. 176]. If we choose $r = 1$ then the same definition of $\langle A_0, A_1 \rangle_{\theta, (r)}$ coincides exactly with the definition of the space, also denoted by $\langle A_0, A_1 \rangle_\theta$, on lines 9–10 of [6, p. 263]. (Cf. also [7, p. 251]. The paper [7] was catalysed in large part by ideas appearing in [14].)

It seems intuitively obvious, as was tacitly assumed in [6] and [7], that choosing the constant r to be $\ln 2$ or 1 , or indeed any other positive number, gives the same space $\langle A_0, A_1 \rangle_\theta$ to within equivalence of norms. In the next section, Theorem 3.6, our main result in this note, will show this explicitly, and will also provide explicit (but probably not optimal) estimates for the constants of equivalence. As already mentioned above, this will be done with the help of an alternative “continuous” definition of these spaces. The existence of this kind of alternative definition answers one case of a question which was already posed in Problème 1.2 of [14, p. 177]. The much better known real and complex methods of interpolation have equivalent “continuous” and “discrete” (or “periodic”) definitions. (See [12, pp. 18–19] or [2, Lemma 3.1.3 p. 41 and Lemma 3.2.3 p. 43] for this result for the real method, and [4, pp. 1007–1009] or [1] for its analogue for the complex method.) So we shall see here, not unexpectedly, that the same is true for the Peetre “plus-minus” method¹.

Another potentially useful fact which will emerge from our discussion here is that the “continuous” version of the “plus-minus” method, which we shall introduce in Section 3 is in some sense the “limit” of its “discrete” versions as r tends to 0 . This will be expressed by the formula (3.7). An analogous result for the complex method is

¹The reason for the name “plus-minus”, which is sometimes used for the method for constructing the spaces $\langle A_0, A_1 \rangle_\theta$, comes from a property of (weakly) unconditionally convergent series which is mentioned, for example, on line 20 of [10, p. 58].

presented in [1]. The analogous result for the real method seems rather obvious. See Remark 4.1 for another role, perhaps its most interesting role, which I expect this "continuous" definition of $\langle A_0, A_1 \rangle_\theta$ to play.

As I already claimed in a previous version of this note, one can surely obtain essentially the same result as our main theorem in an alternative way, by using appropriate modifications of the ideas and methods in [10], in particular, those on p. 59 of that paper. I am very grateful to Mieczysław Mastyło for subsequently indicating, in a private communication, how this can indeed be done.

3. A "CONTINUOUS" DEFINITION OF THE SPACE $\langle A_0, A_1 \rangle_\theta$

Not surprisingly, in this new definition, sums will be replaced by integrals. The role previously played by the set of sequences Λ will now be played by the set, which we will denote by Φ , of all functions $\phi : \mathbb{R} \rightarrow \mathbb{F}$ which satisfy $\sup_{t \in \mathbb{R}} |\phi(t)| \leq 1$ and are locally piecewise continuous, i.e., piecewise continuous on every bounded interval in \mathbb{R} .

For each $\theta \in (0, 1)$ the role of the space $J(\theta, r, A_0, A_1)$ will now be placed by a space which we will denote by $J(\theta, A_0, A_1)$, or simply by J . This will be the space of all functions $u : \mathbb{R} \rightarrow A_0 \cap A_1$ which are locally piecewise continuous and for which, for $j \in \{0, 1\}$, the improper A_j valued Riemann integrals $\int_{-\infty}^{\infty} e^{(j-\theta)t} u(t) dt$ are *unconditionally convergent*, which we will define here to mean that the integrals $\int_{-R}^R e^{(j-\theta)t} \phi(t) u(t) dt$ converge in A_j norm as $R \rightarrow +\infty$ for every choice of $\phi \in \Phi$.

(For our purposes here we do not need to use Bochner integration or other more "advanced" integration methods for Banach space valued functions, since we do not need to consider a more general class of such functions. All integrals appearing here will be "naive" Riemann integrals or improper Riemann integrals of locally piecewise continuous Banach space valued functions, where the relevant Banach space will always be $A_0 \cap A_1$. The range of integration for these intervals, if it is not a bounded or unbounded interval, will be a finite union of such intervals. The definitions and some of the basic properties of such integrals can be recalled e.g., by consulting the appendix of [11, pp. 257–259].)

Remark 3.1. It will be convenient to explicitly state the following obvious "continuous" analogue of the equivalence of conditions (i) and (ii) of Remark 2.4: A function $u : \mathbb{R} \rightarrow A_0 \cap A_1$, which is locally piecewise continuous, is an element of J if and only if, for $j \in \{0, 1\}$ and for each $\varepsilon > 0$ and each $\phi \in \Phi$, there exists a positive number $R(\varepsilon)$ (which could in principle also depend on ϕ and j) such that $\left\| \int_{R_1 \leq |t| \leq R_2} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} \leq \varepsilon$ for all numbers R_1 and R_2 which satisfy $R(\varepsilon) \leq R_1 < R_2$.

The following lemma is a "continuous" analogue of Remark 2.3.

Lemma 3.2. *Suppose that $U \subset V \subset \mathbb{R}$, where the each of the sets U and V is the union of finitely many bounded or unbounded intervals. Suppose that $j \in \{0, 1\}$ and that the function $u : \mathbb{R} \rightarrow A_0 \cap A_1$ is either*

- (i) *an element of J*
- or, alternatively,*

(ii) is a function which is locally piecewise continuous and for which

$$(3.1) \quad \int_V e^{(j-\theta)t} \phi(t) u(t) dt \in A_j \text{ for each } \phi \in \Phi \text{ and } j \in \{0, 1\}.$$

Then, for $j \in \{0, 1\}$,

$$(3.2) \quad \int_U e^{(j-\theta)t} \phi(t) u(t) dt \in A \text{ for each } \phi \in \Phi$$

and

$$(3.3) \quad \sup \left\{ \left\| \int_U e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \leq \sup \left\{ \left\| \int_V e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\}.$$

Proof. For each $\phi \in \Phi$, the form of U and of V ensures that the functions $\phi_U := \phi \chi_U$ and $\phi_V := \phi \chi_V$ are also elements of Φ . In case (ii) the property (3.1) is explicitly imposed. In case (i) the same property follows from the fact that $\int_V e^{(j-\theta)t} \phi(t) u(t) dt = \int_{-\infty}^{\infty} e^{(j-\theta)t} \phi_V(t) u(t) dt \in A_j$ for each $\phi \in \Phi$. This property now implies (3.2) because $\int_U e^{(j-\theta)t} \phi(t) u(t) dt = \int_V e^{(j-\theta)t} \phi_U(t) u(t) dt$ for each $\phi \in \Phi$. This last formula implies that

$$\sup \left\{ \left\| \int_U e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} = \sup \left\{ \left\| \int_V e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi_U \right\}$$

where Φ_U is the set of all functions of the form $\phi_U = \phi \chi_U$ as ϕ ranges over all elements of Φ . Since $\Phi_U \subset \Phi$, we obtain (3.3) and the proof is complete. \square

Note that the preceding simple proof has to be valid, and indeed is valid, even if one or both of the suprema in (3.3) are infinite, since so far we do not know how to exclude that possibility. But the next result will imply that both of these suprema are necessarily finite when $u \in J$.

Fact 3.3. For each $u \in J$,

$$(3.4) \quad \lim_{R \rightarrow +\infty} \sup \left\{ \left\| \int_{|t| \geq R} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} = 0 \text{ for } j \in \{0, 1\}.$$

Proof. (Until we reach the end of this proof, we still cannot, and do not exclude the possibility that $\sup \left\{ \left\| \int_{|t| \geq R} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\}$ could equal ∞ for some or all values of R .)

Suppose that (3.4) does not hold. Then, for at least one value of j and some $\delta > 0$, there exist a strictly increasing unbounded sequence of positive numbers $\{\rho_n\}_{n \in \mathbb{N}}$ and a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in Φ such that $\left\| \int_{|t| \geq \rho_n} e^{(j-\theta)t} \phi_n(t) u(t) dt \right\|_{A_j} \geq \delta$. For each fixed $n \in \mathbb{N}$, since $\int_{-R}^R e^{(j-\theta)t} \phi_n(t) u(t) dt$ converges in A_j norm as $R \rightarrow +\infty$, there exists a number R_n such that $R_n > \rho_n$ and $\left\| \int_{\rho_n \leq |t| < R_n} e^{(j-\theta)t} \phi_n(t) u(t) dt \right\|_{A_j} \geq \delta/2$. Let us recursively define a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that $n_1 = 1$ and $\rho_{n_{k+1}} > R_{n_k}$ for

each $k \geq 1$. Thus the sets $E_k := \{t \in \mathbb{R} : \rho_{n_k} \leq |t| < R_{n_k}\}$ are pairwise disjoint. This means that the function $\psi = \sum_{k \in \mathbb{N}} \phi_{n_k} \chi_{E_k}$ is an element of Φ . So (cf. Remark 3.1) the quantity $\left\| \int_{\rho_{n_k} \leq |t| < R_{n_k}} e^{(j-\theta)t} \psi(t) u(t) dt \right\|_{A_j}$ can be made arbitrarily small when k and therefore ρ_{n_k} are chosen sufficiently large. But this contradicts the fact that, for every $k \in \mathbb{N}$,

$$\left\| \int_{\rho_{n_k} \leq |t| < R_{n_k}} e^{(j-\theta)t} \psi(t) u(t) dt \right\|_{A_j} = \left\| \int_{\rho_{n_k} \leq |t| < R_{n_k}} e^{(j-\theta)t} \phi_{n_k}(t) u(t) dt \right\|_{A_j} \geq \frac{\delta}{2}.$$

We have therefore proved that (3.4) holds. \square

Since $\sup \left\{ \left\| \int_{-n}^n e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \leq \int_{-n}^n e^{(j-\theta)t} \|u(t)\|_{A_j} dt < \infty$ for each $n \in \mathbb{N}$, we deduce from (3.4) that the seminorm

$$(3.5) \quad \|u\|_J := \sup \left\{ \left\| \int_{-\infty}^{\infty} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : j = 0, 1, \phi \in \Phi \right\}$$

is finite for each $u \in J$. The fact that $\|\cdot\|_J$ is not quite a norm on J will not cause any difficulties. (One could of course slightly modify the definition of J , e.g., by adding the requirement that all its elements must be left continuous at all points, so that $\|\cdot\|_J$ would become a norm. But there is no need to do this.)

Remark 3.4. It is clear (analogously to our observation in Remark 2.2) that, for each $u \in J$, the value of the supremum in (3.5) will not change if we replace Φ in (3.5) by its subset Φ_0 consisting of those of its functions which have compact support.

Fact 3.5. For each $u \in J$, the limits $\int_{-\infty}^0 u(t) dt := \lim_{R \rightarrow +\infty} \int_{-R}^0 u(t) dt$ and $\int_0^{\infty} u(t) dt := \lim_{R \rightarrow +\infty} \int_0^R u(t) dt$ exist, the first with respect to the norm of A_0 , and the second with respect to the norm of A_1 . Furthermore they satisfy $\left\| \int_{-\infty}^0 u(t) dt \right\|_{A_0} \leq \|u\|_J$ and $\left\| \int_0^{\infty} u(t) dt \right\|_{A_1} \leq \|u\|_J$. Therefore, for each constant $c \in \mathbb{R}$, the limit $\lim_{R \rightarrow +\infty} \int_{-R}^{R+c} u(t) dt$ exists with respect to the norm of $A_0 + A_1$ and does not depend on c . We denote this limit by $\int_{-\infty}^{\infty} u(t) dt$. It satisfies

$$(3.6) \quad \left\| \int_{-\infty}^{\infty} u(t) dt \right\|_{A_0 + A_1} \leq 2 \|u\|_J.$$

Proof. The function $\phi_0(t) := \chi_{(-\infty, 0]}(t) e^{\theta t}$ is in Φ and therefore, for each $u \in J$, the integrals $\int_{-R}^0 u(t) dt = \int_{-R}^R e^{-\theta t} \phi_0(t) u(t) dt$ converge in A_0 norm as R tends to $+\infty$ and in fact

$$\left\| \int_{-\infty}^0 u(t) dt \right\|_{A_0} = \left\| \int_{-\infty}^{\infty} e^{-\theta t} \phi_0(t) u(t) dt \right\|_{A_0} \leq \|u\|_J.$$

Similarly, the function $\phi_1(t) := \chi_{[0, \infty)}(t)e^{-(1-\theta)t}$ is in Φ and therefore, for each $u \in J$, the integrals $\int_0^R u(t)dt = \int_{-R}^R e^{(1-\theta)t}\phi_1(t)u(t)dt$ converge in A_1 norm as R tends to $+\infty$ and

$$\left\| \int_0^\infty u(t)dt \right\|_{A_1} = \left\| \int_{-\infty}^\infty e^{(1-\theta)t}\phi_1(t)u(t)dt \right\|_{A_1} \leq \|u\|_J.$$

The remaining claims in the statement of Fact 3.5 now follow obviously and immediately. \square

Let $\langle A_0, A_1 \rangle_{\theta, (0)}$ be the space of all elements $a \in A_0 + A_1$ which satisfy $a = \int_{-\infty}^\infty u(t)dt$ for some $u \in J$. We norm this space by $\|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} := \inf \|u\|_J$ where the infimum is taken over all $u \in J$ for which $a = \int_{-\infty}^\infty u(t)dt$. Analogously to some statements above about the spaces $\langle A_0, A_1 \rangle_{\theta, (r)}$ for $r > 0$, here we can use (3.6) to show that this infimum is indeed a norm, rather than merely a seminorm, and also to show that $\langle A_0, A_1 \rangle_{\theta, (0)}$ is continuously embedded in $A_0 + A_1$.

We are now ready to state and prove the main theorem of this note. As an immediate corollary it will provide us with the formula

$$(3.7) \quad \|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} = \lim_{r \searrow 0} \|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} \quad \text{for every } a \in \langle A_0, A_1 \rangle_{\theta, (0)},$$

which motivates our choice of notation $\langle A_0, A_1 \rangle_{\theta, (0)}$ for the space which we have just introduced.

Theorem 3.6. For each Banach couple (A_0, A_1) and each $\theta \in (0, 1)$ and each $r > 0$, the spaces $\langle A_0, A_1 \rangle_{\theta, (r)}$ and $\langle A_0, A_1 \rangle_{\theta, (0)}$ coincide to within equivalence of norms. More explicitly, their norms satisfy

$$(3.8) \quad e^{-r\theta} \|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} \leq \|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} \leq e^{(1-\theta)r} \|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} \quad \forall a \in \langle A_0, A_1 \rangle_{\theta, (0)}.$$

Proof. Our reasoning will be conceptually quite simple, and quite reminiscent of the straightforward arguments which can be used (cf. [12] and [2]) for establishing the "classical" connections between the discrete and continuous J -method constructions of the Lions-Peetre spaces $(A_0, A_1)_{\theta, p}$. However here there are rather more details which have to be carefully checked along the way.

Suppose first that $a \in \langle A_0, A_1 \rangle_{\theta, (r)}$ for some $r > 0$. Given an arbitrary $\varepsilon > 0$, choose a sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $J(\theta, r, A_0, A_1)$, such that $a = \sum_{n=-\infty}^\infty a_n$ and

$$\|\{a_n\}_{n \in \mathbb{Z}}\|_{J(\theta, r, A_0, A_1)} \leq \|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} + \varepsilon.$$

Now let $u : \mathbb{R} \rightarrow A_0 \cap A_1$ be the function defined by

$$(3.9) \quad u = \frac{1}{r} \sum_{n \in \mathbb{Z}} \chi_{[rn, r(n+1))} a_n.$$

For each $\phi \in \Phi$, for each interval $[rn, r(n+1))$ and for j equal to either 0 or 1 we have that

$$\begin{aligned}
\int_{rn}^{r(n+1)} e^{(j-\theta)t} \phi(t) u(t) dt &= \frac{1}{r} e^{(j-\theta)rn} \left(\int_{rn}^{r(n+1)} e^{(j-\theta)(t-rn)} \phi(t) dt \right) a_n \\
(3.10) \quad &= e^{(j-\theta)rn} \mu_n a_n
\end{aligned}$$

where $|\mu_n| \leq \sup \{ |e^{(j-\theta)(t-rn)} \phi(t)| : t \in [rn, r(n+1)] \}$. So, whether $j = 0$ or $j = 1$, we obtain that $|\mu_n| \leq \max \{ e^{(1-\theta)r}, 1 \} = e^{(1-\theta)r}$ for all n .

Let V be a set of the form $V = \bigcup_{n \in F} [rn, r(n+1))$ for some *finite* set F of integers. For $j \in \{0, 1\}$ and each $\phi \in \Phi$ we obtain from (3.10) that

$$(3.11) \quad \int_V e^{(j-\theta)t} \phi(t) u(t) dt = \sum_{n \in F} e^{(j-\theta)rn} \mu_n a_n \in A_j$$

where, as before, we can assert that the numbers μ_n each satisfy

$$(3.12) \quad |\mu_n| \leq e^{(1-\theta)r} \text{ for each } n \in F \text{ and for all choices of } \phi \in \Phi.$$

Let U be an arbitrary union of finitely many arbitrary intervals, but with the additional condition that it must be a subset of V . The sets U and V satisfy the hypotheses of Lemma 3.2. Furthermore, (3.11) shows that u satisfies the condition (ii) of that lemma. So the lemma justifies the transition from the first to the second line in the following calculation. The subsequent transition to the third line follows from (3.11) and (3.12).

$$\begin{aligned}
&\sup \left\{ \left\| \int_U e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \\
&\leq \sup \left\{ \left\| \int_V e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \\
(3.13) \quad &\leq e^{(1-\theta)r} \sup \left\{ \left\| \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}.
\end{aligned}$$

We shall now use some special cases of these estimates to show that the function u is an element of J :

Let ρ and R be arbitrary numbers which satisfy $r < \rho < R$. Let $m(\rho)$ be the unique integer for which

$$(3.14) \quad rm(\rho) < \rho \leq r(m(\rho) + 1)$$

and let $n(R)$ be the unique integer for which $rn(R) > R \geq r(n(R) - 1)$. Then $1 \leq m(\rho) < n(R)$ and

$$\begin{aligned} U : &= \{t \in \mathbb{R} : \rho \leq |t| \leq R\} \subset V := \left\{ t \in \mathbb{R} : |t| \in \bigcup_{m(\rho) \leq k \leq n(R)} [rk, r(k+1)] \right\} \\ &= \bigcup_{m(\rho) \leq k \leq n(R)} ([-r(k+1), -rk] \cup [rk, r(k+1)]) \\ &= \bigcup \{[rk, r(k+1)] : k \in F\} \end{aligned}$$

where F is the union of the two sets $\{k \in \mathbb{Z} : m(\rho) \leq k \leq n(R)\}$ and

$$\{k \in \mathbb{Z} : -n(R) - 1 \leq k \leq -m(\rho) - 1\}.$$

So, in this case, the inequality provided by (3.13) will justify the first two lines of the following calculation. The transition to the third line will be justified by Remark 2.3 together with the fact that F is contained in the set $\{n \in \mathbb{Z} : m(\rho) \leq |n| \leq n(R) + 1\}$.

$$\begin{aligned} &\sup \left\{ \left\| \int_{\rho \leq |t| \leq R} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \\ &\leq e^{(1-\theta)r} \sup \left\{ \left\| \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} \\ &\leq e^{(1-\theta)r} \sup \left\{ \left\| \sum_{m(\rho) \leq |n| \leq n(R)+1} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}. \end{aligned}$$

As an element of $J(\theta, r, A_0, A_1)$, the sequence $\{a_n\}_{n \in \mathbb{Z}}$ must satisfy Condition (iii) of Remark 2.4. I.e., for each $\varepsilon_0 > 0$ there exists an integer $N(\varepsilon_0)$ such that the expression on the last line of the preceding calculation is less than ε_0 whenever $m(\rho) > N(\varepsilon_0)$. Therefore, using (3.14), we deduce that the supremum in the first line of the preceding calculation is less than ε_0 whenever $\rho \geq r(N(\varepsilon_0) + 1)$. Since ε_0 is arbitrary this suffices to show (cf. Remark 3.1) that $u \in J$. (For this we of course also need u to be locally piecewise continuous. But that is obvious from (3.9).)

The fact that $u \in J$ implies in turn (cf. Fact 3.5) that the integral $\int_{-\infty}^{\infty} u(t) dt$ exists as an element of $A_0 + A_1$ equal to the limit in $A_0 + A_1$ norm as $R \rightarrow +\infty$ of the integrals $\int_{-R}^R u(t) dt$. This limit must coincide with

$$\lim_{n \rightarrow \infty} \int_{-rn}^{rn} u(t) dt = \lim_{n \rightarrow \infty} \sum_{k=-n}^{n-1} \int_{rk}^{r(k+1)} u(t) dt = \lim_{n \rightarrow \infty} \sum_{k=-n}^{n-1} a_k = \lim_{n \rightarrow \infty} \sum_{k=-n}^n a_k = a.$$

(Here we have used the result described just before (2.5) and chosen the constant k_0 appearing there to equal -1 .) We also claim that

$$(3.15) \quad \|u\|_J \leq e^{(1-\theta)r} \|\{a_n\}_{n \in \mathbb{Z}}\|_{J(\theta, r, (A_0, A_1))}.$$

To show this, it will suffice (cf. Remark 3.4) to show that

$$(3.16) \quad \sup \left\{ \left\| \int_U e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\} \leq e^{(1-\theta)r} \left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, (A_0, A_1))}$$

for $j \in \{0, 1\}$ and for each bounded interval $U \subset \mathbb{R}$. Each such U can of course be contained in a set of the form $V = \bigcup_{n \in F} [rn, r(n+1))$ for some finite set F of integers. So we can apply the inequalities (3.13) together with the fact (cf. Remark 2.3) that

$$\begin{aligned} & \sup \left\{ \left\| \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} \\ & \leq \sup \left\{ \left\| \sum_{n \in \mathbb{Z}} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\} \leq \left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, (A_0, A_1))} \end{aligned}$$

to obtain (3.16) for each such U and therefore also (3.15). Thus we have shown that $a \in \langle A_0, A_1 \rangle_{\theta, (0)}$ and $\|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} \leq \|u\|_J \leq e^{(1-\theta)r} (\|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} + \varepsilon)$. Since a is an arbitrary element of $\langle A_0, A_1 \rangle_{\theta, (r)}$ and since ε is an arbitrary positive number, we conclude that $\langle A_0, A_1 \rangle_{\theta, (r)}$ is continuously embedded in $\langle A_0, A_1 \rangle_{\theta, (0)}$ and that the norms of these two spaces satisfy the second inequality in (3.8).

Now suppose, conversely, that $a \in \langle A_0, A_1 \rangle_{\theta, (0)}$. Given an arbitrary $\varepsilon > 0$, let u be an element of J for which $a = \int_{-\infty}^{\infty} u(t) dt$ and $\|u\|_J \leq \|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} + \varepsilon$. Define the sequence $\{a_n\}_{n \in \mathbb{Z}}$ of elements of $A_0 \cap A_1$ by $a_n = \int_{rn}^{r(n+1)} u(t) dt$ for each $n \in \mathbb{Z}$. Given an arbitrary sequence $\{\lambda_k\}_{k \in \mathbb{Z}}$ in Λ , define a function $\psi \in \Phi$ associated with $\{\lambda_k\}_{k \in \mathbb{Z}}$ by setting $\psi = \sum_{k \in \mathbb{Z}} \lambda_k \chi_{[rk, r(k+1))}$. Then, for every finite subset F of \mathbb{Z} , and for $j \in \{0, 1\}$, we have

$$\begin{aligned} \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n &= \sum_{n \in F} \int_{rn}^{r(n+1)} e^{(j-\theta)rn} \psi(t) u(t) dt \\ &= \sum_{n \in F} \int_{rn}^{r(n+1)} e^{(j-\theta)t} \xi_j(t) u(t) dt, \end{aligned}$$

where $\xi_j(t) = \sum_{n \in F} e^{(j-\theta)(rn-t)} \psi(t) \chi_{[rn, r(n+1))}$. It is clear that the function ξ_j is locally piecewise continuous and that, for all $t \in \mathbb{R}$, we have $|\xi_0(t)| \leq e^{r\theta}$ and $|\xi_1(t)| \leq 1$. So $e^{-r\theta} \xi_j \in \Phi$ for both values of j . Therefore, for every $\{\lambda_k\}_{k \in \mathbb{Z}}$ in Λ and every finite set $F \subset \mathbb{Z}$, we have

$$(3.17) \quad \left\| \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} \leq e^{r\theta} \sup \left\{ \left\| \int_{E_F} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\},$$

where

$$(3.18) \quad E_F = \bigcup_{n \in F} [rn, r(n+1)).$$

Let us now specialize to the case where F is a set of the form $F = \{n \in \mathbb{Z} : n_1 \leq |n| \leq n_2\}$ for integers n_1 and n_2 which satisfy $n_2 > n_1 > 2$. Then E_F is contained in $\{t \in \mathbb{R} : |t| \geq R_1\}$ whenever the number R_1 satisfies

$$(3.19) \quad 0 < R_1 < r(n_1 - 1),$$

and, since $u \in J$, we can apply Lemma 3.2 (case (i)) with $U = E_F$ and $V = \{t \in \mathbb{R} : R_1 \leq |t|\}$ to deduce from (3.17) that

$$(3.20) \quad \left\| \sum_{n_1 \leq |n| \leq n_2} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} \leq e^{r\theta} \sup \left\{ \left\| \int_{|t| \geq R_1} e^{(j-\theta)t} \phi(t) u(t) dt \right\|_{A_j} : \phi \in \Phi \right\}$$

In view of Fact 3.3, the right side of (3.20) can be made arbitrarily small by choosing R_1 sufficiently large. So, in view of (3.19), the left side of (3.20) is arbitrarily small whenever n_1 is sufficiently large. Since this property holds for $j = 0, 1$ and for each $\{\lambda_k\}_{k \in \mathbb{Z}}$ in Λ , we see (cf. Remark 2.4) that the sequence $\{a_n\}_{n \in \mathbb{Z}}$ is an element of $J(\theta, r, A_0, A_1)$. Therefore the sum $\sum_{n=-\infty}^{\infty} a_n$ is an element of $\langle A_0, A_1 \rangle_{\theta, (r)}$ which is the limit in $A_0 + A_1$ as N tends to ∞ of the partial sums

$$\sum_{n=-N}^N a_n = \sum_{n=-N}^N \int_{rn}^{r(n+1)} u(t) dt = \int_{-rN}^{r(N+1)} u(t) dt.$$

So this limit equals $\int_{-\infty}^{\infty} u(t) dt = a$. (Here we have used Fact 3.5, with the constant c appearing there now chosen to equal r .)

In order to estimate the norm of a in $\langle A_0, A_1 \rangle_{\theta, (r)}$, we once again apply Lemma 3.2 (case (i)) to (3.17) and (3.18), but this time with $U = E_F$ for an *arbitrary* finite subset F of \mathbb{Z} and with $V = \mathbb{R}$. This gives that $\left\| \sum_{n \in F} e^{(j-\theta)rn} \lambda_n a_n \right\|_{A_j} \leq e^{r\theta} \|u\|_J$ for $j \in \{0, 1\}$ and every $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda$, independently of our choice of F . Therefore (cf. Remark 2.2) we have that $\left\| \{a_n\}_{n \in \mathbb{Z}} \right\|_{J(\theta, r, A_0, A_1)} \leq e^{r\theta} \|u\|_J$ and, consequently,

$$\|a\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} \leq e^{r\theta} \left(\|a\|_{\langle A_0, A_1 \rangle_{\theta, (0)}} + \varepsilon \right).$$

This completes the proof that $\langle A_0, A_1 \rangle_{\theta, (0)}$ is continuously embedded in $\langle A_0, A_1 \rangle_{\theta, (r)}$ and that the norms of these two spaces satisfy the first inequality in (3.8). Therefore this also completes the proof of the theorem. \square

Remark 3.7. Of course the inequalities (3.8) can be used to obtain estimates for the equivalence constants between the norms $\|\cdot\|_{\langle A_0, A_1 \rangle_{\theta, (r_1)}}$ and $\|\cdot\|_{\langle A_0, A_1 \rangle_{\theta, (r_2)}}$ for any two positive numbers r_1 and r_2 , should anyone ever need them. But these are very unlikely to be optimal estimates, given that they are obtained via passage through the norm $\|\cdot\|_{\langle A_0, A_1 \rangle_{\theta, (0)}}$. One can probably obtain better estimates fairly easily in the case where $r_1 = 2r_2$ or r_1 is some other integer multiple of r_2 .

4. THE INCLUSION $\langle A_0, A_1 \rangle_\theta \subset [A_0, A_1]_\theta$

Already in 1971 (see the formula (1.2) on p. 176 of [14]), Jaak Peetre remarked that the space $\langle A_0, A_1 \rangle_\theta$ is continuously embedded in the complex interpolation space

$[A_0, A_1]_\theta$ introduced by Alberto Calderón in [3]. (In fact the remarks in [14] show that $\langle A_0, A_1 \rangle_\theta$ is contained in a possibly smaller "periodic/discrete" variant of $[A_0, A_1]_\theta$ which would later be shown (see [4]) to coincide with $[A_0, A_1]_\theta$.) For the reader's convenience, in Theorem 4.3 of this section, we explicitly present the simple proof that $\langle A_0, A_1 \rangle_\theta \subset [A_0, A_1]_\theta$. We make a point of noting that this inclusion is a norm one embedding, and we also briefly mention another norm one estimate in terms of "periodic" norms on $[A_0, A_1]_\theta$.

Remark 4.1. The inclusion $\langle A_0, A_1 \rangle_\theta \subset [A_0, A_1]_\theta$ is the easy part of the proof of another interesting property of the space $\langle A_0, A_1 \rangle_\theta$, namely that it *coincides* with $[A_0, A_1]_\theta$, to within equivalence of norms, whenever (A_0, A_1) is a couple of complexified Banach lattices of measurable functions on the same underlying measure space. This property has sometimes proved useful, for example in [6], and I hope to use it further in the very near future, in a forthcoming paper (which in fact has been the main motivation for me to write this note). I also conjecture that this coincidence to within equivalence of norms of $\langle A_0, A_1 \rangle_\theta$ and $[A_0, A_1]_\theta$ for couples of lattices is even an *isometry* when $\langle A_0, A_1 \rangle_\theta$ is equipped with the norm of $\langle A_0, A_1 \rangle_{\theta,(0)}$, i.e., the norm associated with the apparently new "continuous" method for its construction presented in Section 3. Hence my interest in explicitly noting that the embedding $\langle A_0, A_1 \rangle_{\theta,(r)} \subset [A_0, A_1]_\theta$ has norm one.

Remark 4.2. The contents of the preceding remark make it almost compulsory to mention an example due to Svante Janson [10, Example 6, p. 62]. This example can be used to show that $\langle A_0, A_1 \rangle_\theta \neq [A_0, A_1]_\theta$ for some Banach couples (A_0, A_1) which do not have the above mentioned lattice structure. More explicitly, if (A_0, A_1) is the couple (FL_0^1, FL_1^1) (in the notation of [5, p. 81]), then the above-mentioned Example 6, combined with Theorems 3 and 5 on pages 57 and 59 of [10] shows that the sequence space $\langle FL_0^1, FL_1^1 \rangle_\theta$ is contained in the weighted ℓ^2 space $\left\{ \{\lambda_n\}_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |e^{\theta n} \lambda_n|^2 < \infty \right\}$. On the other hand, as shown in the proof of Theorem 22 on p. 68 of [10], $[FL_0^1, FL_1^1]_\theta$ contains all complex sequences $\{\lambda_n\}_{n \in \mathbb{Z}}$ for which $\{e^{\theta n} \lambda_n\}_{n \in \mathbb{Z}}$ is the sequence of Fourier coefficients of some function in $L^1(\mathbb{T})$. Since $L^2(\mathbb{T})$ is strictly smaller than $L^1(\mathbb{T})$ this shows that $\langle FL_0^1, FL_1^1 \rangle_\theta$ is strictly smaller than $[FL_0^1, FL_1^1]_\theta$.

Theorem 4.3. (Cf. [14, (1.2) p. 176] and [10].) *For every Banach couple (A_0, A_1) of complex Banach spaces, and for each $r \geq 0$ and $\theta \in (0, 1)$, the inclusion*

$$\langle A_0, A_1 \rangle_{\theta,(r)} \subset [A_0, A_1]_\theta$$

holds, and

$$(4.1) \quad \|b\|_{[A_0, A_1]_\theta} \leq \|b\|_{\langle A_0, A_1 \rangle_{\theta,(r)}} \quad \text{for every } b \in \langle A_0, A_1 \rangle_{\theta,(r)}.$$

Proof. It suffices to treat the case where $r > 0$, since the case where $r = 0$ will then follow immediately from Theorem 3.6 and (3.7).

Given an arbitrary sequence $\{a_n\}_{n \in \mathbb{Z}}$ of elements of $A_0 \cap A_1$, let $b_U = \sum_{n \in U} a_n$ for every finite subset U of \mathbb{Z} . Obviously b_U is an element of $[A_0, A_1]_\theta$ since it is an element of $A_0 \cap A_1$. But we want to control its norm in $[A_0, A_1]_\theta$. For each $\delta > 0$, we introduce

the function $f_\delta(z) = e^{\delta(z-\theta)^2} \sum_{n \in U} e^{(z-\theta)rn} a_n$ which is clearly an element of Calderón's space $\mathcal{F}(A_0, A_1)$ and satisfies $f_\delta(\theta) = b_U$ and also

$$\begin{aligned} \|f_\delta\|_{\mathcal{F}(A_0, A_1)} &= \sup \left\{ \|f_\delta(j + it)\|_{A_j} : j \in \{0, 1\}, t \in \mathbb{R} \right\} \\ &\leq e^\delta \sup \left\{ \left\| \sum_{n \in U} e^{(j+it-\theta)rn} a_n \right\|_{A_j} : j \in \{0, 1\}, t \in \mathbb{R} \right\} \\ &\leq e^\delta \sup \left\{ \left\| \sum_{n \in U} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : j \in \{0, 1\}, \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}. \end{aligned}$$

Since δ can be chosen arbitrarily small, we deduce that

$$(4.2) \quad \|b_U\|_{[A_0, A_1]_\theta} \leq \sup \left\{ \left\| \sum_{n \in U} \lambda_n e^{(j-\theta)rn} a_n \right\|_{A_j} : j \in \{0, 1\}, \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda \right\}.$$

Let b be an arbitrary element of $\langle A_0, A_1 \rangle_{\theta, (r)}$ and let ε be an arbitrary positive number. Then there exists a sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $J(\theta, r, A_0, A_1)$ with $\|\{a_n\}_{n \in \mathbb{Z}}\|_{J(\theta, r, A_0, A_1)} \leq (1 + \varepsilon) \|b\|_{\langle A_0, A_1 \rangle_{\theta, (r)}}$ and such that $b = \sum_{n \in \mathbb{Z}} a_n$ (convergence in $A_0 + A_1$). As above, let us define $b_U = \sum_{n \in U} a_n$ for every finite subset U of \mathbb{Z} . Since $\{a_n\}_{n \in \mathbb{Z}}$ necessarily satisfies Condition (iii) of Remark 2.4, we can deduce from (4.2) that, for each $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $\|b_U\|_{[A_0, A_1]_\theta} < \varepsilon$ whenever $U \subset \{n \in \mathbb{Z} : |n| \geq N(\varepsilon)\}$. This shows that the sequence $\{s_k\}_{k \in \mathbb{N}}$ defined by $s_k = \sum_{n=-k}^k a_n$ is a Cauchy sequence in $[A_0, A_1]_\theta$. Since this sequence converges in $A_0 + A_1$ to b , this means that $b \in [A_0, A_1]_\theta$ and also that $\|b\|_{[A_0, A_1]_\theta} = \lim_{k \rightarrow \infty} \|s_k\|_{[A_0, A_1]_\theta}$. Letting U_k be the set of integers n satisfying $|n| \leq k$, we use (4.2) again, and then (2.4), with $U = U_k$ and $V = \mathbb{Z}$, to obtain that

$$\|s_k\|_{[A_0, A_1]_\theta} = \|b_{U_k}\|_{[A_0, A_1]_\theta} \leq \|\{a_n\}_{n \in \mathbb{Z}}\|_{J(\theta, r, A_0, A_1)} \leq (1 + \varepsilon) \|b\|_{\langle A_0, A_1 \rangle_{\theta, (r)}}.$$

Since ε is arbitrary, this completes the proof. \square

Remark 4.4. For each finite set $U \subset \mathbb{Z}$, the function $f(z) = \sum_{n \in U} e^{(z-\theta)rn} a_n$ satisfies $f(z + 2\pi i/r)$ for all $z \in \mathbb{C}$. Therefore, if we choose $\lambda = 2\pi/r$, this function is an element of the "periodic" space $\mathcal{F}_\lambda(A_0, A_1)$ as defined on p. 1007 of [4]. (Cf. also [1, pp. 161–162].) Therefore, for each $r > 0$, the same reasoning as in the proof of the previous theorem in fact gives a slightly stronger estimate than (4.1), since it is expressed in terms of the larger "periodic" norm $\|\cdot\|_{[A_0, A_1]_\theta^\lambda}$, defined on that same page of [4]. I.e., we obtain that

$$\|b\|_{[A_0, A_1]_\theta^{2\pi/r}} \leq \|b\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} \quad \text{for all } r > 0, \theta \in (0, 1) \text{ and } b \in \langle A_0, A_1 \rangle_{\theta, (r)}.$$

5. APPENDIX - COMPLETENESS OF THE SPACES $J(\theta, r, A_0, A_1)$ AND $\langle A_0, A_1 \rangle_\theta$

For the convenience of those readers who may happen to be less familiar with these kinds of topics, here is a proof, via more or less routine arguments, of the completeness

of $J(\theta, r, A_0, A_1)$ and, consequently, of $\langle A_0, A_1 \rangle_{\theta(r)}$ for each $r > 0$ and $\theta \in (0, 1)$. The completeness of $\langle A_0, A_1 \rangle_{\theta(r)}$ for $r = 0$ can then be deduced immediately from Theorem 3.6. (So we see that the obvious fact that the space J is not complete nor even normed, does not create any difficulties.) Naturally enough, the fact that $\langle A_0, A_1 \rangle_\theta$ is complete was noticed and mentioned already in [14].

We begin by considering two-sided sequences $\{a_n\}_{n \in \mathbb{Z}}$ which take values in an arbitrary Banach space A . As in Example 2.4 of [7, p. 247], we let $UC(A)$ denote the space of all such sequences $\{a_n\}_{n \in \mathbb{Z}}$ for which $\sum_{n \in \mathbb{Z}} a_n$ is unconditionally convergent in A . As already remarked on p. 2, an A valued sequence $\{a_n\}_{n \in \mathbb{Z}}$ is in $UC(A)$ if and only if it satisfies the condition (2.1). Since A is complete, (2.1) is equivalent, in turn, to

$$(5.1) \quad \lim_{N \rightarrow \infty} \sup \left\{ \left\| \sum_{|n| \geq N} \lambda_n a_n \right\|_A : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda_0 \right\} = 0.$$

Condition (5.1) ensures that the functional

$$\|\{a_n\}_{n \in \mathbb{Z}}\|_{UC(A)} := \sup \left\{ \left\| \sum_{n \in \mathbb{Z}} \lambda_n a_n \right\|_A : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda_0 \right\}$$

is finite for each $\{a_n\}_{n \in \mathbb{Z}} \in UC(A)$ and therefore defines a norm² on $UC(A)$. Note that this norm satisfies

$$(5.2) \quad \|a_{n_0}\|_A \leq \|\{a_n\}_{n \in \mathbb{Z}}\|_{UC(A)} \quad \text{for each fixed } n_0 \in \mathbb{Z} \text{ and each } \{a_n\}_{n \in \mathbb{Z}} \in UC(A).$$

The main step of our proof will be to show that $UC(A)$ is complete with respect to this norm. (This will confirm, as asserted on p. 247 of [7], that the mapping UC is indeed a "pseudolattice" (cf. Definition 2.1 of [7, p. 246]).)

For each sequence $a = \{a_n\}_{n \in \mathbb{Z}}$ in $UC(A)$ let Ta be the numerical sequence $\{(Ta)_N\}_{N \in \mathbb{Z}}$ defined by

$$(Ta)_N = \sup \left\{ \left\| \sum_{|n| \geq N} \lambda_n a_n \right\|_A : \{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda_0 \right\}.$$

This defines a (nonlinear) map $T : UC(A) \rightarrow c_0$ which satisfies

$$(5.3) \quad \|Ta - Tb\|_{\ell^\infty} \leq \|a - b\|_{UC(A)}$$

for all pairs of elements $a = \{a_n\}_{n \in \mathbb{Z}}$ and $b = \{b_n\}_{n \in \mathbb{Z}}$ in $UC(A)$. Now suppose that $\{a_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $UC(A)$. For each fixed $m \in \mathbb{N}$, the element a_m is a two-sided A valued sequence which we will denote by $\{a_{m,n}\}_{n \in \mathbb{Z}}$ and Ta_m is a one-sided numerical sequence which we shall denote by $\{(Ta_m)_k\}_{k \in \mathbb{N}}$. It follows from (5.3) that the sequence $\{Ta_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in c_0 . Therefore the same kinds of standard arguments used to characterize the relatively compact subsets of c_0 show that the numbers $\rho_N := \sup \{|(Ta_m)_k| : k \geq N, m \in \mathbb{N}\}$ satisfy $\lim_{N \rightarrow \infty} \rho_N = 0$.

In view of (5.2), for each fixed $n_0 \in \mathbb{Z}$, the sequence $\{a_{m,n_0}\}_{m \in \mathbb{N}}$ is Cauchy in A and therefore there exists an A valued sequence $b = \{b_n\}_{n \in \mathbb{Z}}$ such that $\lim_{m \rightarrow \infty} \|a_{m,n} - b_n\|_A =$

²This norm is different from, but equivalent to the norm chosen for this space in [7].

0 for each fixed $n \in \mathbb{Z}$. For each $N \in \mathbb{N}$ and each $\{\lambda_k\}_{k \in \mathbb{Z}} \in \Lambda_0$, we have $\left\| \sum_{|n| \geq N} \lambda_n b_n \right\|_A = \lim_{m \rightarrow \infty} \left\| \sum_{|n| \geq N} \lambda_n a_{m,n} \right\|_A \leq \rho_N$. This shows see that the sequence b has the property (5.1) and is therefore an element of $UC(A)$.

Given any $\varepsilon > 0$, let us choose $N = N(\varepsilon)$ sufficiently large so that $\rho_N < \varepsilon/4$. Then, choose $M = M(\varepsilon)$ such that $\|a_{m,n} - b_n\|_A \leq \varepsilon/4N$ for each integer $m \geq M$ and each integer $n \in [-N, N]$. These choices of N and M ensure that, for each $m \geq M$, and each $\{\lambda_k\}_{k \in \mathbb{Z}}$ in Λ_0 ,

$$\begin{aligned} \left\| \sum_{n \in \mathbb{Z}} \lambda_n (a_{m,n} - b_n) \right\|_A &\leq \left\| \sum_{n=-N+1}^{N-1} \lambda_n (a_{m,n} - b_n) \right\|_A + \left\| \sum_{|n| \geq N} \lambda_n (a_{m,n} - b_n) \right\|_A \\ &\leq \sum_{n=-N+1}^{N-1} \|a_{m,n} - b_n\| + \left\| \sum_{|n| \geq N} \lambda_n a_{m,n} \right\|_A + \left\| \sum_{|n| \geq N} \lambda_n b_n \right\|_A \\ &\leq (2N-1) \frac{\varepsilon}{4N} + \rho_N + \rho_N < \varepsilon. \end{aligned}$$

Taking the supremum as $\{\lambda_k\}_{k \in \mathbb{Z}}$ ranges over all of Λ_0 , we obtain that $\|b - a_m\|_{UC(A)} \leq \varepsilon$ for all $m \geq M$. This completes the proof that $UC(A)$ is complete.

The rest of the proof of the completeness of $J(\theta, r, A_0, A_1)$ and then of $\langle A_0, A_1 \rangle_\theta$ is a special case of the argument indicated immediately after Definition 2.11 on p. 249 of [7]. More explicitly, we now observe that the space $J(\theta, r, A_0, A_1)$ is the intersection of a "weighted" version of $UC(A_0)$ with a differently "weighted" version of $UC(A_1)$. So the completeness of $J(\theta, r, A_0, A_1)$ is a simple consequence of the completeness of each of the two spaces $UC(A_0)$ and $UC(A_1)$ and the fact that A_0 and A_1 are both continuously embedded in the Banach space $A_0 + A_1$. The completeness of $\langle A_0, A_1 \rangle_\theta$ can be very easily and "routinely" deduced from the completeness of $J(\theta, r, A_0, A_1)$, e.g., by using that completeness to show that, for any sequence $\{x_m\}_{m \in \mathbb{N}}$ in $\langle A_0, A_1 \rangle_{\theta, (r)}$ which satisfies $\sum_{m=1}^{\infty} \|x_m\|_{\langle A_0, A_1 \rangle_{\theta, (r)}} < \infty$ there exists an element $y \in \langle A_0, A_1 \rangle_\theta$ such that $\lim_{N \rightarrow \infty} \left\| y - \sum_{m=1}^N x_m \right\|_{\langle A_0, A_1 \rangle_\theta} = 0$.

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